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Representations of supersymmetry algebras with Dirac bispinor generators

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Abstract. Explicit formulae for linear and nonlinear realizations of the supersymmetry groups with Dirac bispinor generators are given and Cartan forms are obtained.

1. Introduction

Supersymmetry algebras introduced recently by several authors (Wess and Zumino 1974, Volkov and Akulov 1973, Gel'fand and Lichtman 1972) bring together bosons and fermions into irreducible multiplets. This remarkable property results from the fact that the supersymmetry algebra contains both commutators and anticommutators. The supersymmetry algebra considered by Wess and Zumino (1974), and from another point of view by Volkov and Akulov (1973), has the following form:

$$\begin{aligned}&\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\sigma_{\mu\alpha\dot\beta}P^\mu\\ &\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0\\ &[Q_\alpha, P_\mu] = 0\end{aligned} \tag{1}$$

where Q_α is the two-component spinor. It is interesting that this algebra is the only possible non-trivial extension of Poincaré algebra by the two-component spinor Q.

All possible extensions of the Poincaré algebra by a four-component bispinor W have been investigated by Gel'fand and Lichtman (1972). They found that there are seven possibilities

$A^\pm$ $\quad [P_\mu, W] = \Pi_\pm\gamma_\mu W, \qquad [P_\mu, \bar{W}] = -\bar{W}\Pi_\pm\gamma_\mu,$

$\quad \{W, W\} = \{\bar{W}, \bar{W}\} = 0, \qquad \{W, \bar{W}\} = \Pi_\pm(\gamma P),$

$B^\pm$ $\quad [P_\mu, W] = \Pi_\pm\gamma_\mu W, \qquad [P_\mu, \bar{W}] = -\bar{W}\Pi_\pm\gamma_\mu,$

$\quad \{W, W\} = \{\bar{W}, \bar{W}\} = 0, \qquad \{W, \bar{W}\} = 0,$

$C^\pm$ $\quad [P_\mu, W] = 0, \qquad [P_\mu, \bar{W}] = 0,$

$\quad \{W, W\} = \{\bar{W}, \bar{W}\} = 0, \qquad \{W, \bar{W}\} = \Pi_\pm(\gamma P),$

D $\quad [P_\mu, W] = 0, \qquad [P_\mu, \bar{W}] = 0,$

$\quad \{W, W\} = \{\bar{W}, \bar{W}\} = 0, \qquad \{W, \bar{W}\} = \gamma P,$

where†

$$\Pi_{\pm}=\tfrac{1}{2}(1\pm\gamma_5).$$

In the present paper in § 2 we investigate the linear, and in § 3 the nonlinear, realizations of the supersymmetry group with generators obeying the Gel'fand–Lichtman algebras.

2. Linear representations

2.1. General remarks

To obtain the linear representations of the Gel'fand–Lichtman supersymmetry we use the beautiful method of Salam and Strathdee (1974) extended by Ferrara *et al* (1974). The representations of the supersymmetry group G are induced from those of the Lorentz group L. We are interested in the action of the generators P_μ and W on a superfield $\phi(x, \theta, \bar{\theta})$ defined on a cosets space (superspace), where the components of the constant bispinors θ and $\bar{\theta}=\theta^\dagger\gamma_0$ are the generating elements of the Grassman algebra.

Let us start with some remarks.

(i) The algebras $A^{\pm}$, $B^{\pm}$, $C^{\pm}$ are not invariant under space inversion.

(ii) The algebras $A^{\pm}$, $B^{\pm}$, and $C^{\pm}$ contain anticommutators of the form $\{F, F^\dagger\}=0$, which implies an indefinite metric in the space of states‡.

(iii) Re-defining $P_\mu\to P_\mu$, $W\to R^{-1}W$ and letting R tend to infinity we see that the algebras $B^{\pm}$ are the contractions of the algebras $A^{\pm}$. Putting $P_\mu\to R^{-1}P_\mu$, $W\to R^{-\frac{1}{2}}W$ we can show in a similar manner that the algebras $C^{\pm}$ are the contractions of the $A^{\pm}$ too.

(iv) The algebras $C^{\pm}$ and D contain the algebra of Wess and Zumino as a subalgebra.

(v) The algebras $A^{\pm}$, $B^{\pm}$ and $C^{\pm}$ have ideals generated by $W_{\mp}=\Pi_{\mp}W$ and $\bar{W}_{\mp}=\bar{W}\Pi_{\pm}$. The quotient algebras $P\oplus W\oplus\bar{W}/W_{\mp}\oplus\bar{W}_{\mp}$ are isomorphic to the Wess–Zumino algebra in the cases $A^{\pm}$ and $C^{\pm}$, and to the trivial one

$$\{W_\pm, W_\pm\}=\{\bar{W}_\pm, \bar{W}_\pm\}=\{W_\pm, \bar{W}_\pm\}=0 \qquad [P_\mu, W_\pm]=0$$

in the cases $B^{\pm}$. So the supersymmetry constraints $W_{\mp}\phi=\bar{W}_{\mp}\phi=0$ imply representations of the quotient algebra. The constraints $W_{\mp}\phi=0$ and $\bar{W}_{\mp}\phi=0$ have, in all parametrizations, the forms

$$(\Pi_{\mp})_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}\phi=0 \qquad \text{and} \qquad (\Pi_{\pm})_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi=0$$

respectively.

(vi) A superfield can be expanded in a power series in θ_α and $\bar{\theta}_{\dot{\alpha}}$

$$\phi(x,\theta,\bar{\theta})=\sum_{k,r}\phi^{[\alpha_1\ldots\alpha_k][\dot{\beta}_1\ldots\dot{\beta}_r]}\theta_{\alpha_1}\ldots\theta_{\alpha_k}\bar{\theta}_{\dot{\beta}_1}\ldots\bar{\theta}_{\dot{\beta}_r}.$$

† We use the Dirac matrices with the following properties: $\{\gamma_\mu,\gamma_\nu\}=2\eta_{\mu\nu}$, $\gamma^\dagger=-\gamma$, $\gamma_0^\dagger=\gamma_0$, $\gamma_5^\dagger=\gamma_5$, $\gamma_5^2=1$, $\eta_{\mu\nu}=\mathrm{diag}(+---)$.

‡ We are grateful to Professor J T Lopuszanski for this remark.

This expansion has a finite number of terms because the Grassman algebra generated by θ_α and $\bar\theta_{\dot\alpha}$ is finite. Fields $\phi(x)^{[\alpha_1\ldots\alpha_k][\dot\beta_1\ldots\dot\beta_r]}$ are antisymmetric in indices $[\alpha_1\ldots\alpha_k]$ and $[\dot\beta_1\ldots\dot\beta_r]$. The number of independent components of $\phi(x)^{[\alpha_1\ldots\alpha_k][\dot\beta_1\ldots\dot\beta_r]}$ equals $\binom{4}{k}\binom{4}{r}$. These fields are generally reducible. If the superfield $\phi(x,\theta,\bar\theta)$ transforms according to the representation D of the Lorentz group then the coefficients $\phi^{[\alpha_1\ldots\alpha_k][\dot\beta_1\ldots\dot\beta_r]}$ belong to the representation $D\otimes D_k\otimes\bar D_r$, where

$$\begin{aligned}
&D_0(\bar D_0)\sim D^{00}\\
&D_1(\bar D_1)\sim D^{\frac12 0}\oplus D^{0\frac12}\\
&D_2(\bar D_2)\sim D^{\frac12\frac12}\oplus 2D^{00}\\
&D_3(\bar D_3)\sim D^{\frac12 0}\oplus D^{0\frac12}\\
&D_4(\bar D_4)\sim D^{00}.
\end{aligned}\tag{3}$$

Imposing on a superfield in the cases $A^\pm$, $B^\pm$ and $C^\pm$ one of the conditions $W_\mp\phi=0$ or $\bar W_\mp\phi=0$ one finds that D_k or $\bar D_r$ reduces to D'_k or $\bar D'_r$, where

$$\begin{aligned}
&D'_0\sim D^{00} &&\bar D'_0\sim D^{00}\\
&D'_1\sim D^{\frac12 0} &&\bar D'_1\sim D^{0\frac12}\\
&D'_2\sim D^{00} &&\bar D'_2\sim D^{00}\\
&D'_3\sim D^{\frac12 0} &&\bar D'_3\sim D^{0\frac12}\\
&D'_4\sim D^{00} &&\bar D'_4\sim D^{00}.
\end{aligned}\tag{4}$$

2.2 *Representations of the algebras A*

We can choose three different parametrizations of the superspace

$$\exp[\mathrm{i}(xP+\bar W\theta+\bar\theta W)]=\phi_0(x,\theta,\bar\theta)\tag{5a}$$

$$\exp[\mathrm{i}(xP)]\exp[\mathrm{i}(\bar W\theta+\bar\theta W)]=\phi_1(x,\theta,\bar\theta)\tag{5b}$$

$$\exp[\mathrm{i}(\bar W\theta+\bar\theta W)]\exp[\mathrm{i}(xP)]=\phi_2(x,\theta,\bar\theta).\tag{5c}$$

These superfields are connected by the relations

$$\phi_1(x,\theta,\bar\theta)=\phi_0(x,\theta-\tfrac12\mathrm{i}\Pi_\pm(\gamma x)\theta,\bar\theta+\tfrac12\mathrm{i}\bar\theta\Pi_\pm(\gamma x))\tag{6a}$$

$$\phi_2(x,\theta,\bar\theta)=\phi_0(x,\theta+\tfrac12\mathrm{i}\Pi_\pm(\gamma x)\theta,\bar\theta-\tfrac12\mathrm{i}\bar\theta\Pi_\pm(\gamma x)).\tag{6b}$$

The action of the translation and the supersymmetry transformation on the superfields is

$$\exp(\mathrm{i}aP)\phi_0(x,\theta,\bar\theta)=\phi_0(x+a,(1-\tfrac12\mathrm{i}\Pi_\pm(\gamma a))\theta,\bar\theta(1+\tfrac12\mathrm{i}\Pi_\pm(\gamma a)))$$

$$\begin{aligned}
&\exp[\mathrm{i}(\bar\zeta W+\bar W\zeta)]\phi_0(x,\theta,\bar\theta)\\
&\quad=\phi_0(x_\mu+\tfrac12\mathrm{i}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta),\ \theta+(1+\tfrac12\mathrm{i}\Pi_\pm(\gamma x))\zeta-\tfrac1{12}\Pi_\pm\gamma^\mu(\zeta-\theta)\\
&\qquad\times(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta),\\
&\qquad\bar\theta+\bar\zeta(1-\tfrac12\mathrm{i}\Pi_\pm(\gamma x))+\tfrac1{12}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)(\bar\zeta-\bar\theta)\Pi_\pm\gamma^\mu)
\end{aligned}\tag{7a}$$

$$\exp(\mathrm{i}aP)\phi_1(x,\theta,\bar\theta)=\phi_1(x+a,\theta,\bar\theta)$$

$$\begin{aligned}\exp[\mathrm{i}(\bar\zeta W+\bar W\zeta)]\phi_1(x,\theta,\bar\theta) &= \phi_1(x_\mu+\tfrac{1}{2}\mathrm{i}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta),\ \theta+(1+\mathrm{i}\Pi_\pm(\gamma x))\zeta-\Pi_\pm\gamma^\mu(\tfrac{1}{3}\zeta+\tfrac{1}{6}\theta)\\ &\quad\times(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta),\\ &\quad\bar\theta+\bar\zeta(1-\mathrm{i}\Pi_\pm(\gamma x))+(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)(\tfrac{1}{3}\bar\zeta+\tfrac{1}{6}\bar\theta)\Pi_\pm\gamma^\mu)\end{aligned} \tag{7b}$$

$$\exp(\mathrm{i}aP)\phi_2(x,\theta,\bar\theta)=\phi_2(x+a,(1-\mathrm{i}\Pi_\pm(\gamma a))\theta,\bar\theta(1+\mathrm{i}\Pi_\pm(\gamma a))$$

$$\begin{aligned}\exp[\mathrm{i}(\bar\zeta W+\bar W\zeta)]\phi_2(x,\theta,\bar\theta) &= \phi_2(x_\mu+\tfrac{1}{2}\mathrm{i}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta),\ \theta+\zeta+\Pi_\pm\gamma^\mu(\tfrac{1}{3}\theta+\tfrac{1}{6}\zeta)\\ &\quad\times(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta),\\ &\quad\bar\theta+\bar\zeta-(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)(\tfrac{1}{6}\bar\zeta+\tfrac{1}{3}\bar\theta)\Pi_\pm\gamma^\mu)\end{aligned} \tag{7c}$$

or, in the infinitesimal form

$$\delta_a\phi_0=\left(a_\mu\frac{\partial}{\partial x_\mu}-\tfrac{1}{2}\mathrm{i}\Pi_\pm(\gamma a)\theta\frac{\partial}{\partial\theta}+\tfrac{1}{2}\mathrm{i}\bar\theta\Pi_\pm(\gamma a)\frac{\partial}{\partial\bar\theta}\right)\phi_0$$

$$\begin{aligned}\delta_\zeta\phi_0=\Big[&\tfrac{1}{2}\mathrm{i}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)\frac{\partial}{\partial x_\mu}+((1+\tfrac{1}{2}\mathrm{i}\Pi_\pm(\gamma x))\zeta+\tfrac{1}{12}\Pi_\pm\gamma^\mu\theta(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta))\frac{\partial}{\partial\theta}\\ &+(\bar\zeta(1-\tfrac{1}{2}\mathrm{i}\Pi_\pm(\gamma x))-\tfrac{1}{12}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)\bar\theta\Pi_\pm\gamma^\mu)\frac{\partial}{\partial\bar\theta}\Big]\phi_0\end{aligned} \tag{8a}$$

$$\delta_a\phi_1=\left(a_\mu\frac{\partial}{\partial x_\mu}\right)\phi_1$$

$$\begin{aligned}\delta_\zeta\phi_1=\Big[&\tfrac{1}{2}\mathrm{i}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)\frac{\partial}{\partial x_\mu}+((1+\mathrm{i}\Pi_\pm(\gamma x))\zeta-\Pi_\pm\gamma^\mu\tfrac{1}{6}\theta(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta))\frac{\partial}{\partial\theta}\\ &+(\bar\zeta(1-\mathrm{i}\Pi_\pm(\gamma x))+(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)\tfrac{1}{6}\bar\theta\Pi_\pm\gamma^\mu)\frac{\partial}{\partial\bar\theta}\Big]\phi_1\end{aligned} \tag{8b}$$

$$\delta_a\phi_2=\left(a_\mu\frac{\partial}{\partial x_\mu}-\mathrm{i}\Pi_\pm(\gamma a)\theta\frac{\partial}{\partial\theta}+\mathrm{i}\bar\theta\Pi_\pm(\gamma a)\frac{\partial}{\partial\bar\theta}\right)\phi_2$$

$$\begin{aligned}\delta_\zeta\phi_2=\Big[&\tfrac{1}{2}\mathrm{i}(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)\frac{\partial}{\partial x_\mu}+(\zeta+\Pi_\pm\gamma^\mu\tfrac{1}{3}\theta(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta))\frac{\partial}{\partial\theta}\\ &+(\bar\zeta-(\bar\zeta\Pi_\pm\gamma_\mu\theta-\bar\theta\Pi_\pm\gamma_\mu\zeta)\tfrac{1}{3}\bar\theta\Pi_\pm\gamma^\mu)\frac{\partial}{\partial\bar\theta}\Big]\phi_2.\end{aligned} \tag{8c}$$

In these cases no simple constraints beyond those discussed above, i.e. $W_\mp\phi=0$, $\bar W_\mp\phi=0$, can be imposed.

2.3 Representations of the algebras $B^\pm$

In this case three different parametrizations (equations (5*a*), (5*b*) and (5*c*)) can also be chosen. The superfields ϕ_0, ϕ_1 and ϕ_2 are connected by relations (6*a*) and (6*b*). The action of the group elements on the superfields is

$$\exp(iaP)\phi_0(x,\theta,\bar\theta)=\phi_0(x+a,(1-\tfrac{1}{2}i\Pi_\pm(\gamma a))\theta,\bar\theta(1+\tfrac{1}{2}i\Pi_\pm(\gamma a)))$$

$$\exp[i(\bar\zeta W+\bar W\zeta)]\phi_0(x,\theta,\bar\theta)=\phi_0(x,\theta+(1+\tfrac{1}{2}i\Pi_\pm(\gamma x))\zeta,\bar\theta+\bar\zeta(1-\tfrac{1}{2}i\Pi_\pm(\gamma x))) \quad (9a)$$

$$\exp(iaP)\phi_1(x,\theta,\bar\theta)=\phi_1(x+a,\theta,\bar\theta)$$

$$\exp[i(\bar\zeta W+\bar W\zeta)]\phi_1(x,\theta,\bar\theta)=\phi_1(x,\theta+(1+i\Pi_\pm(\gamma x))\zeta,\bar\theta+\bar\zeta(1-i\Pi_\pm(\gamma x))) \quad (9b)$$

$$\exp(iaP)\phi_2(x,\theta,\bar\theta)=\phi_2(x+a,(1-i\Pi_\pm(\gamma a))\theta,\bar\theta(1+i\Pi_\pm(\gamma a)))$$

$$\exp[i(\bar\zeta W+\bar W\zeta)]\phi_2(x,\theta,\bar\theta)=\phi_2(x,\theta+\zeta,\bar\theta+\bar\zeta) \quad (9c)$$

or, in the infinitesimal form

$$\delta_a\phi_0=\left(a_\mu\frac{\partial}{\partial x_\mu}-\tfrac{1}{2}i\Pi_\pm(\gamma a)\theta\frac{\partial}{\partial\theta}+\tfrac{1}{2}i\bar\theta\Pi_\pm(\gamma a)\frac{\partial}{\partial\bar\theta}\right)\phi_0$$

$$\delta_\zeta\phi_0=\left((1+\tfrac{1}{2}i\Pi_\pm(\gamma x))\zeta\frac{\partial}{\partial\theta}+\bar\zeta(1-\tfrac{1}{2}i\Pi_\pm(\gamma x))\frac{\partial}{\partial\bar\theta}\right)\phi_0 \quad (10a)$$

$$\delta_a\phi_1=\left(a_\mu\frac{\partial}{\partial x_\mu}\right)\phi_1$$

$$\delta_\zeta\phi_1=\left((1+i\Pi_\pm(\gamma x))\zeta\frac{\partial}{\partial\theta}+\bar\zeta(1-i\Pi_\pm(\gamma x))\frac{\partial}{\partial\bar\theta}\right)\phi_1 \quad (10b)$$

$$\delta_a\phi_2=\left(a_\mu\frac{\partial}{\partial x_\mu}-i\Pi_\pm(\gamma a)\theta\frac{\partial}{\partial\theta}+i\bar\theta\Pi_\pm(\gamma a)\frac{\partial}{\partial\bar\theta}\right)\phi_2$$

$$\delta_\zeta\phi_2=\left(\zeta\frac{\partial}{\partial\theta}+\bar\zeta\frac{\partial}{\partial\bar\theta}\right)\phi_2. \quad (10c)$$

Because θ and $\bar\theta$ enter in the differential operators (equation (10)) only as coefficients before $\partial/\partial\theta$ and $\partial/\partial\bar\theta$ respectively, the conditions $\partial\phi/\partial\theta=0$ and $\partial\phi/\partial\bar\theta=0$ are supersymmetry-invariant constraints. If these constraints are imposed we have

$$(\Pi_\pm)_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi=0,\qquad(\Pi_\mp)_{\alpha\beta}\frac{\partial}{\partial\bar\theta_\beta}\phi=0,\qquad(\Pi_\mp)_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi=0,\qquad(\Pi_\pm)_{\alpha\beta}\frac{\partial}{\partial\bar\theta_\beta}\phi=0.$$

The first two constraints are the same as in the case $A^\pm$. We can show that from the other two it follows again that $\partial\phi/\partial\theta=0$ and $\partial\phi/\partial\bar\theta=0$. Indeed, let us consider, for example, the third parametrization. Then $W_\pm$ and $\bar W_\pm$ act as $(\Pi_\pm)_{\alpha\beta}(\partial/\partial\bar\theta_\beta)$ and $(\Pi_\mp)_{\alpha\beta}(\partial/\partial\theta_\alpha)$. From the commutation rules

$$[P_\mu,W_\pm]=\gamma_\mu W_\mp,\qquad[P_\mu,\bar W_\pm]=-\bar W_\mp\gamma_\mu \quad (11)$$

and from the constraints under consideration, $W_\pm\phi=0$ and $\bar W_\pm\phi=0$, one obtains $W_\mp\phi=0$ and $\bar W_\mp\phi=0$ respectively. Thus

$$\partial\phi/\partial\theta\equiv W\phi=(W_\pm+W_\mp)\phi=0$$

and

$$\partial\phi/\partial\bar\theta\equiv\bar W\phi=(\bar W_\pm+\bar W_\mp)\phi=0.$$

The same can be shown with small modification for the other parametrizations. Finally, we have the following possibilities for the constraints:

$$(\Pi_{\pm})_{\alpha\beta}\frac{\partial}{\partial\theta_{\alpha}}\phi=0 \tag{12a}$$

$$(\Pi_{\mp})_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_{\beta}}\phi=0 \tag{12b}$$

$$\frac{\partial}{\partial\theta}\phi=0 \tag{12c}$$

$$\frac{\partial}{\partial\bar{\theta}}\phi=0. \tag{12d}$$

In accord with these, various representations of the Lorentz group occur in the superfield expansion (see table 1).

Table 1. The Lorentz transformation properties of superfield components.

Constraints	Algebra B^+	Algebra B^-
(12*a*)	$D\otimes\bar{D}'_k\otimes\bar{D}_r$	$D\otimes D'_k\otimes\bar{D}_r$
(12*b*)	$D\otimes D_k\otimes\bar{D}'_r$	$D\otimes D_k\otimes D'_r$
(12*c*)	$D\otimes\bar{D}_r$	$D\otimes\bar{D}_r$
(12*d*)	$D\otimes D_k$	$D\otimes D_k$
(12*a*) and (12*b*)	$D\otimes\bar{D}'_k\otimes\bar{D}'_r$	$D\otimes D'_k\otimes D'_r$
(12*a*) and (12*d*)	$D\otimes\bar{D}'_k$	$D\otimes D'_k$
(12*b*) and (12*c*)	$D\otimes\bar{D}'_r$	$D\otimes D'_r$

2.4 *Representations of the algebras* $C^{\pm}$

In this case, because of the commutation rules $[P_{\mu}, W]=[P_{\mu}, \bar{W}]=0$, the possible parametrizations are

$$\exp[i(xP+\bar{\theta}W+\bar{W}\theta)]=\phi_0(x,\theta,\bar{\theta}) \tag{13a}$$

$$\exp[i(xP+\bar{\theta}W)]\exp(i\bar{W}\theta)=\phi_1(x,\theta,\bar{\theta}) \tag{13b}$$

$$\exp[i(xP+\bar{W}\theta)]\exp(i\bar{\theta}W)=\phi_2(x,\theta,\bar{\theta}). \tag{13c}$$

The superfields are connected by the relations

$$\begin{aligned}\phi_1(x,\theta,\bar{\theta})&=\phi_0(x_{\mu}+\tfrac{1}{2}i\bar{\theta}\Pi_{\pm}\gamma_{\mu}\theta,\theta,\bar{\theta})\\ \phi_2(x,\theta,\bar{\theta})&=\phi_0(x_{\mu}-\tfrac{1}{2}i\bar{\theta}\Pi_{\pm}\gamma_{\mu}\theta,\theta,\bar{\theta}).\end{aligned} \tag{14}$$

The group element action is

$$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]\phi_0(x,\theta,\bar{\theta})=\phi_0(x_{\mu}+\tfrac{1}{2}i(\bar{\zeta}\Pi_{\pm}\gamma_{\mu}\theta-\bar{\theta}\Pi_{\pm}\gamma_{\mu}\zeta),\theta+\zeta,\bar{\theta}+\bar{\zeta}) \tag{15a}$$

$$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]\phi_1(x,\theta,\bar{\theta})=\phi_1(x_{\mu}-i\bar{\theta}\Pi_{\pm}\gamma_{\mu}\zeta-\tfrac{1}{2}i\bar{\zeta}\Pi_{\pm}\gamma_{\mu}\zeta,\theta+\zeta,\bar{\theta}+\bar{\zeta}) \tag{15b}$$

$$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]\phi_2(x,\theta,\bar{\theta})=\phi_2(x_{\mu}+i\bar{\zeta}\Pi_{\pm}\gamma_{\mu}\theta+\tfrac{1}{2}i\bar{\zeta}\Pi_{\pm}\gamma_{\mu}\zeta,\theta+\zeta,\bar{\theta}+\bar{\zeta}) \tag{15c}$$

and in all three cases $\exp(iaP)\phi(x, \theta, \bar{\theta}) = \phi(x+a, \theta, \bar{\theta})$. In the infinitesimal form we have

$$\delta_\zeta\phi_0 = \left(\tfrac{1}{2}\mathrm{i}(\bar{\zeta}\Pi_\pm\gamma_\mu\theta - \bar{\theta}\Pi_\pm\gamma_\mu\zeta)\frac{\partial}{\partial x_\mu} + \zeta\frac{\partial}{\partial\theta} + \bar{\zeta}\frac{\partial}{\partial\bar{\theta}}\right)\phi_0 \tag{16a}$$

$$\delta_\zeta\phi_1 = \left(-\mathrm{i}\bar{\theta}\Pi_\pm\gamma_\mu\zeta\frac{\partial}{\partial x_\mu} + \zeta\frac{\partial}{\partial\theta} + \bar{\zeta}\frac{\partial}{\partial\bar{\theta}}\right)\phi_1 \tag{16b}$$

$$\delta_\zeta\phi_2 = \left(\mathrm{i}\bar{\zeta}\Pi_\pm\gamma_\mu\theta\frac{\partial}{\partial x_\mu} + \zeta\frac{\partial}{\partial\theta} + \bar{\zeta}\frac{\partial}{\partial\bar{\theta}}\right)\phi_2. \tag{16c}$$

The form of constraints depends here on the choice of parametrization. From equations (16b) and (16c) it follows immediately that the constraints $\partial\phi_1/\partial\theta = 0$ and $\partial\phi_2/\partial\bar{\theta} = 0$ are invariant. Of course, they are no longer implied by the conditions $\partial(\Pi_\mp)_{\alpha\beta}\phi_1/\partial\theta_\alpha = 0$ or $\partial(\Pi_\pm)_{\alpha\beta}\phi_2/\partial\bar{\theta}_\beta = 0$. From equation (14) we see that for every parametrization four conditions can be written, namely

$$(\Pi_\pm)_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi_0 = 0, \qquad (\Pi_\mp)_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}\phi_0 = 0$$

$$\left(\frac{\partial}{\partial\theta_\alpha} - \tfrac{1}{2}\mathrm{i}(\bar{\theta}\Pi_\pm\gamma_\mu)_\alpha\frac{\partial}{\partial x_\mu}\right)(\Pi_\mp)_{\alpha\beta}\phi_0 = 0$$

$$(\Pi_\pm)_{\alpha\beta}\left(\frac{\partial}{\partial\bar{\theta}_\beta} - \tfrac{1}{2}\mathrm{i}(\Pi_\pm\gamma_\mu\theta)_\beta\frac{\partial}{\partial x_\mu}\right)\phi_0 = 0 \tag{17a}$$

$$(\Pi_\pm)_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi_1 = 0, \qquad (\Pi_\mp)_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}\phi_1 = 0$$

$$(\Pi_\mp)_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi_1 = 0, \qquad (\Pi_\pm)_{\alpha\beta}\left(\frac{\partial}{\partial\bar{\theta}_\beta} - \mathrm{i}(\Pi_\pm\gamma_\mu\theta)_\beta\frac{\partial}{\partial x_\mu}\right)\phi_1 = 0 \tag{17b}$$

$$(\Pi_\pm)_{\alpha\beta}\frac{\partial}{\partial\theta_\alpha}\phi_2 = 0, \qquad (\Pi_\mp)_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}\phi_2 = 0$$

$$(\Pi_\pm)_{\alpha\beta}\frac{\partial}{\partial\bar{\theta}_\beta}\phi_2 = 0, \qquad \left(\frac{\partial}{\partial\theta_\alpha} - \mathrm{i}(\bar{\theta}\Pi_\pm\gamma_\mu)_\alpha\frac{\partial}{\partial x_\mu}\right)(\Pi_\mp)_{\alpha\beta}\phi_2 = 0. \tag{17c}$$

The conditions (17) impose various restrictions on representations of the Lorentz group to which components of superfields belong, or they imply connections between these components.

2.5. *Representations of the algebra D*

The possible parametrizations are the same as in the case $C^\pm$. The superfields are connected by the relations

$$\phi_1(x, \theta, \bar{\theta}) = \phi_0(x_\mu + \tfrac{1}{2}\mathrm{i}\bar{\theta}\gamma_\mu\theta, \theta, \bar{\theta})$$
$$\phi_2(x, \theta, \bar{\theta}) = \phi_0(x_\mu - \tfrac{1}{2}\mathrm{i}\bar{\theta}\gamma_\mu\theta, \theta, \bar{\theta}). \tag{18}$$

The group element action is

$$\exp[\mathrm{i}(\bar{\zeta}W+\bar{W}\zeta)]\phi_0(x,\theta,\bar{\theta})=\phi_0(x_\mu+\tfrac{1}{2}\mathrm{i}(\bar{\zeta}\gamma_\mu\theta-\bar{\theta}\gamma_\mu\zeta),\theta+\zeta,\bar{\theta}+\bar{\zeta}) \tag{19a}$$

$$\exp[\mathrm{i}(\bar{\zeta}W+\bar{W}\zeta)]\phi_1(x,\theta,\bar{\theta})=\phi_1(x_\mu-\mathrm{i}\bar{\theta}\gamma_\mu\zeta-\tfrac{1}{2}\mathrm{i}\bar{\zeta}\gamma_\mu\zeta,\theta+\zeta,\bar{\theta}+\bar{\zeta}) \tag{19b}$$

$$\exp[\mathrm{i}(\bar{\zeta}W+\bar{W}\zeta)]\phi_2(x,\theta,\bar{\theta})=\phi_2(x_\mu+\mathrm{i}\bar{\zeta}\gamma_\mu\theta+\tfrac{1}{2}\mathrm{i}\bar{\zeta}\gamma_\mu\zeta,\theta+\zeta,\bar{\theta}+\bar{\zeta}) \tag{19c}$$

or, in the infinitesimal form

$$\delta_\zeta\phi_0=\left(\tfrac{1}{2}\mathrm{i}(\bar{\zeta}\gamma_\mu\theta-\bar{\theta}\gamma_\mu\zeta)\frac{\partial}{\partial x_\mu}+\zeta\frac{\partial}{\partial\theta}+\bar{\zeta}\frac{\partial}{\partial\bar{\theta}}\right)\phi_0 \tag{20a}$$

$$\delta_\zeta\phi_1=\left(-\mathrm{i}\bar{\theta}\gamma_\mu\zeta\frac{\partial}{\partial x_\mu}+\zeta\frac{\partial}{\partial\theta}+\bar{\zeta}\frac{\partial}{\partial\bar{\theta}}\right)\phi_1 \tag{20b}$$

$$\delta_\zeta\phi_2=\left(\mathrm{i}\bar{\zeta}\gamma_\mu\theta\frac{\partial}{\partial x_\mu}+\zeta\frac{\partial}{\partial\theta}+\bar{\zeta}\frac{\partial}{\partial\bar{\theta}}\right)\phi_2. \tag{20c}$$

The constraints invariant under both supersymmetry transformations and space inversion are

$$\left(\frac{\partial}{\partial\theta_\alpha}-\tfrac{1}{2}\mathrm{i}(\bar{\theta}\gamma_\mu)_\alpha\frac{\partial}{\partial x_\mu}\right)\phi_0=0,\qquad\left(\frac{\partial}{\partial\bar{\theta}_\alpha}-\tfrac{1}{2}\mathrm{i}(\gamma_\mu\theta)_\alpha\frac{\partial}{\partial x_\mu}\right)\phi_0=0 \tag{21a}$$

$$\frac{\partial}{\partial\theta}\phi_1=0,\qquad\left(\frac{\partial}{\partial\bar{\theta}_\alpha}-\mathrm{i}(\gamma_\mu\theta)_\alpha\frac{\partial}{\partial x_\mu}\right)\phi_1=0 \tag{21b}$$

$$\frac{\partial}{\partial\bar{\theta}}\phi_2=0,\qquad\left(\frac{\partial}{\partial\theta_\alpha}-\mathrm{i}(\bar{\theta}\gamma_\mu)_\alpha\frac{\partial}{\partial x_\mu}\right)\phi_2=0. \tag{21c}$$

With the help of the projectors $\Pi_\pm$ one can obtain the chirial form of the constraints.

3. Nonlinear realizations and Cartan forms

3.1. General remarks

As is well known, nonlinear realizations of the group G which are linear on the subgroup H are generated by action G on the coset space G/H. Let G be the supersymmetry group and H the Lorentz group L. Then G/L is superspace. The action of G on the superspace has been determined in § 2.

Let the superspace be parametrized by x_μ and bispinor fields $\psi(x)$ and $\bar{\psi}(x)$. Because the supersymmetry transformations must preserve the Hermitian character of x_μ, not all parametrizations are admissible.

Nonlinear realizations and Cartan forms corresponding to the Gel'fand-Lichtman algebras are given below.

3.2. The algebras $A^\pm$

All three parametrizations (5a), (5b), (5c) are allowed. Correspondingly we have

$\exp(\mathrm{i}aP)$:

$$x_\mu\to x'_\mu=x_\mu+a_\mu,\qquad\psi(x)\to\psi'(x')=(1-\tfrac{1}{2}\mathrm{i}\alpha^2\Pi_\pm(\gamma a))\psi(x)$$

$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]$:

$$x_\mu \to x'_\mu = x_\mu + \tfrac{1}{2}i\lambda^2(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta)$$

$$\psi(x)\to\psi'(x') = \psi(x) + (1+\tfrac{1}{2}i\alpha^2\Pi_\pm(\gamma x))\zeta - \tfrac{1}{12}(\alpha\lambda)^3\Pi_\pm\gamma_\mu(\zeta-\psi(x)) \times(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta) \tag{22a}$$

$\exp(iaP)$:

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \qquad \psi(x)\to\psi'(x') = \psi(x)$$

$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]$:

$$x_\mu \to x'_\mu = x_\mu + \tfrac{1}{2}i\lambda^2(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta)$$

$$\psi(x)\to\psi'(x') = \psi(x) + (1+i\alpha^2\Pi_\pm(\gamma x))\zeta - (\alpha\lambda)^3\Pi_\pm\gamma^\mu(\tfrac{1}{3}\zeta+\tfrac{1}{6}\psi(x)) \times(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta) \tag{22b}$$

$\exp(iaP)$:

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \qquad \psi(x)\to\psi'(x') = (1-i\alpha^2\Pi_\pm(\gamma a))\psi(x)$$

$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]$:

$$x_\mu \to x'_\mu = x_\mu + \tfrac{1}{2}i\lambda^2(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta)$$

$$\psi(x)\to\psi'(x') = \psi(x) + \zeta + (\alpha\lambda)^3\Pi_\pm\gamma^\mu(\tfrac{1}{6}\zeta+\tfrac{1}{3}\psi(x)) \times(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta) \tag{22c}$$

where α and λ are dimensional constants. The Cartan forms are

$$\omega^P_\mu = dx_\mu + \tfrac{1}{2}i\lambda^2(d\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\, d\psi(x))$$

$$\omega^\psi = d\psi(x) + \tfrac{1}{2}i\alpha^2\Pi_\pm((\gamma x)\, d\psi(x) - (\gamma\, dx)\psi(x)) + \tfrac{1}{12}(\alpha\lambda)^3(d\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\, d\psi(x))\Pi_\pm\gamma^\mu\psi(x) \tag{23a}$$

$$\omega^P_\mu = dx_\mu + \tfrac{1}{2}i\lambda^2(d\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\, d\psi(x))$$

$$\omega^\psi = d\psi(x) - i\alpha^2\Pi_\pm(\gamma\, dx)\psi(x) + \tfrac{1}{12}(\alpha\lambda)^3(d\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\, d\psi(x))\Pi_\pm\gamma^\mu\psi(x) \tag{23b}$$

$$\omega^P_\mu = dx_\mu + \tfrac{1}{2}i\lambda^2(d\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\, d\psi(x))$$

$$\omega^\psi = (1+i\alpha^2\Pi_\pm(\gamma x))\, d\psi(x) + \tfrac{1}{12}(\alpha\lambda)^3(d\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\, d\psi(x))\Pi_\pm\gamma^\mu\psi(x). \tag{23c}$$

3.3. The algebras $B^\pm$

The same parametrizations as in the case of algebras $A^\pm$ can be considered, so we have $\exp(iaP)$:

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \qquad \psi(x)\to\psi'(x') = (1-\tfrac{1}{2}i\alpha^2\Pi_\pm(\gamma a))\psi(x)$$

$\exp[i(\bar{\zeta}W+\bar{W}\zeta)]$:

$$x_\mu \to x'_\mu = x_\mu, \qquad \psi(x)\to\psi'(x') = \psi(x) + (1+\tfrac{1}{2}i\alpha^2\Pi_\pm(\gamma x))\zeta \tag{24a}$$

exp(iaP):

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \qquad \psi(x) \to \psi'(x') = \psi(x)$$

exp[i($\bar{\zeta}W + \bar{W}\zeta$)]:

$$x_\mu \to x'_\mu = x_\mu, \qquad \psi(x) \to \psi'(x') = \psi(x) + (1 + \mathrm{i}\alpha^2 \Pi_\pm(\gamma x))\zeta \tag{24b}$$

exp(iaP):

$$x_\mu \to x'_\mu = x_\mu + a_\mu, \qquad \psi(x) \to \psi'(x') = \psi(x) + (1 + \mathrm{i}\alpha^2 \Pi_\pm(\gamma a))\psi(x)$$

exp[i($\bar{\zeta}W + \bar{W}\zeta$)]:

$$x_\mu \to x'_\mu = x_\mu, \qquad \psi(x) \to \psi'(x') = \psi(x) + \zeta. \tag{24c}$$

The Cartan forms are

$$\omega^P_\mu = \mathrm{d}x_\mu$$
$$\omega^\psi = \mathrm{d}\psi(x) + \tfrac{1}{2}\mathrm{i}\alpha^2 \Pi_\pm((\gamma x)\,\mathrm{d}\psi(x) - (\gamma\,\mathrm{d}x)\psi(x)) \tag{25a}$$

$$\omega^P_\mu = \mathrm{d}x_\mu$$
$$\omega^\psi = \mathrm{d}\psi(x) - \mathrm{i}\alpha^2 \Pi_\pm(\gamma\,\mathrm{d}x)\psi(x) \tag{25b}$$

$$\omega^P_\mu = \mathrm{d}x_\mu$$
$$\omega^\psi = (1 + \mathrm{i}\alpha^2 \Pi_\pm(\gamma x))\,\mathrm{d}\psi(x). \tag{25c}$$

3.4. *The algebras $C^\pm$*

Because of the commutation rules $[P_\mu, W] = [P_\mu, \bar{W}] = 0$ we have essentially one admissible parametrization (equation (13a)). We obtain

exp[i($\bar{\zeta}W + \bar{W}\zeta$)]:

$$x_\mu \to x'_\mu = x_\mu + \tfrac{1}{2}\mathrm{i}\lambda^2(\bar{\zeta}\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\zeta), \qquad \psi(x) \to \psi'(x') = \psi(x) + \zeta. \tag{26}$$

Translations act in the standard fashion. The Cartan forms are

$$\omega^P_\mu = \mathrm{d}x_\mu + \tfrac{1}{2}\mathrm{i}\lambda^2(\mathrm{d}\bar{\psi}(x)\Pi_\pm\gamma_\mu\psi(x) - \bar{\psi}(x)\Pi_\pm\gamma_\mu\,\mathrm{d}\psi(x))$$
$$\omega^\psi = \mathrm{d}\psi(x). \tag{27}$$

3.5. *The algebra D*

The admissible parametrization is the same as in the cases $C^\pm$. We have

exp[i($\bar{\zeta}W + \bar{W}\zeta$)]:

$$x_\mu \to x'_\mu = x_\mu + \tfrac{1}{2}\mathrm{i}\lambda^2(\bar{\zeta}\gamma_\mu\psi(x) - \bar{\psi}(x)\gamma_\mu\zeta)$$
$$\psi(x) \to \psi'(x') = \psi(x) + \zeta \tag{28}$$

with the standard action of translations. The Cartan forms are

$$\omega^P_\mu = \mathrm{d}x_\mu + \tfrac{1}{2}\mathrm{i}\lambda^2(\mathrm{d}\bar{\psi}(x)\gamma_\mu\psi(x) - \bar{\psi}(x)\gamma_\mu\,\mathrm{d}\psi(x))$$
$$\omega^\psi = \mathrm{d}\psi(x). \tag{29}$$

4. Remark

We wish to make one comment about Lagrangians which are invariant under the action of the transformations described above. In the linear case, the existence of the renormalizable Lagrangian which is invariant under the Wess–Zumino supersymmetry transformations suggests the existence of such Lagrangians at least in the cases $C^{\pm}$ and D. In the nonlinear case, dynamics is introduced by the use of Cartan forms (Volkov and Akulov 1973). The resulting theories are non-renormalizable.

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